Three Graph Booklets will be needed for this subject

WEEK 1
PARTIAL FRACTION

In our earlier classes, we learnt how to add and subtract arithmetic and algebraic fraction. Example

$$\frac{2}{x+5}+\frac{1}{x-4}$$

$$\begin{aligned}\therefore \frac{2(x-4)+x+5}{(x+5)(x+-4)} &= \frac{2x-8+x+5}{(x+5)(x+-4)}\\ &= \frac{3x-3}{(x+5)(x+-4)}\\ &= \frac{3(x-1)}{(x+5)(x+-4)}\end{aligned}$$

When we add or subtract two fractions, we form a bigger or smaller single fraction.

In mathematics it is sometimes neccesary to break a single algebraic fraction down to smaller fractions called partial fractions. In this chapter, we will study the various processes involved in breaking down single fractions into partial fractions

Principle of Undetermined Coefficient

This states that if two polynomials are identically equal, we may equate the coefficients of like powers of the variable.

For example, if

$$2x^4-3x^3+5x^2+3 = Ax^4+Bx^3-Cx^2+Dx+E$$

Then $A=2$; $B=-3$; $-C=5 \Longrightarrow C=-5$; $D=0$ and $E=3$

Example 1.1
If $2x^2+3x-5 = Ax^2-Bx+C$ what is the value of A, B and C

Solution

$$2x^2+3x-5 = Ax^2-Bx+C$$

$\therefore \quad A=2;\ -B=3 \quad \Longrightarrow \quad B=-3$ and $C=-5$

Example1.2
If $x^2-3x+7 = A(x-1)(x+2)+B(x-3)+C$, find A, B, C

Solution

$$\begin{aligned}x^2-3x+7 &= A(x-1)(x+2)+B(x-3)+C\\ &= A(x^2+2x-x-2)+Bx-3B+C\\ &= A(x^2+x-2)+Bx-3B+C\\ &= Ax^2+Ax+Bx-2A-3B+C\\ &= Ax^2+(A+B)x-2A-3B+C\end{aligned}$$

$$\therefore x^2-3x+7 = Ax^2+(A+B)x-2A-3B+C$$

then

$A=1 \qquad (1)$

$A+B=-3 \qquad (2)$

$-2A-3B+C=7 \qquad (3)$

substituting the value of A in (2) we have

$$1+B=-3$$

substituting the values of A and B in (3) we obtain

$$\begin{aligned}-2(1)-3(-4)+C &= 7\\ -2+12+C &= 7\\ -10+C &= 7\\ C &= 7-10\\ C &= -3\end{aligned}$$

$\therefore \quad A=1;\ B=-4$ and $C=-3$

Exercises A

1. If $x^2-5x+9 = A(x+3)(x-2)+B(x-1)-C$, find the values of A, B and C
2. The expansion $A(x+1)(x-3)+B(x-5)(x+2)+C(x-1)+C(c-3)(x-2)$ has a cosntant value for all values of x. Determine the values of A, B and C
3. If $x^3-6x^2+11x+9 = (x+2)(ax^2+bx+c)$, find the values of a, b and c

Addition/ subtraction of "partial fraction" Converting into single fraction

Consider adding $\frac{1}{2}$ and $\frac{1}{3}$

$$\frac{1}{2}+\frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}$$

Hence we say that $\frac{1}{2}$ and $\frac{1}{3}$ are partial fractions of $\frac{5}{6}$

Again consider adding $\frac{1}{x+1}$ and $\frac{2}{x-3}$

$$\begin{aligned}\frac{1}{x+1}+\frac{2}{x-3} &= \frac{x-3+2(x+1)}{(x+1)(x-3)}\\ &= \frac{x-3+2x+2}{(x+1)(x-3)}\\ &= \frac{3x-1}{(x+1)(x-3)}\end{aligned}$$

Hence $\frac{1}{x+1}$ and $\frac{2}{x-3}$ are called the partial fractions of $\frac{3x-1}{(x+1)(x-3)}$

Example 1.3

Simplify $\dfrac{2}{x+1} - \dfrac{1}{x+2}$.

Solution

$$\begin{aligned}\frac{2}{x+1} - \frac{1}{x+2} &= \frac{2(x+2)-(x+1)}{(x+1)(x+2)}\\ &= \frac{2x+4-x-1}{(x+1)(x+2)}\\ &= \frac{x+3}{(x+1)(x+2)}\end{aligned}$$

This means that $\dfrac{2}{x+1}$ and $-\dfrac{1}{x+2}$ are partial fractions of $\dfrac{x+3}{(x+1)(x+2)}$

Example 1.4
Simplify $\dfrac{4}{5x+4} + \dfrac{4}{x+1} - \dfrac{3}{2(x+3)}$

Solution

$$\begin{aligned}\frac{4}{5x+4} + \frac{4}{x+1} - \frac{3}{2(x+3)} &= \frac{4(x+1)(2x+6)+4(5x+4)(2x+6)-3(5x+4)(x+1)}{(5x+4)(x+1)(2x+6)}\\ &= \frac{4(2x^2+8x+6)+4(10x^2+38x+24)-3(5x^2+9x+4)}{(5x+4)(x+1)(2x+6)}\\ &= \frac{8x^2+32x+24+40x^2+152x+96-15x^2-27x-27}{(5x+4)(x+1)(2x+6)}\\ &= \frac{33x^2+157x+93}{(5x+4)(x+1)(2x+6)}\end{aligned}$$

Hence $\dfrac{4}{5x+1}$, $\dfrac{4}{x+1}$ and $-\dfrac{3}{2(x+3)}$ are said to be the partial fraction of $\dfrac{33x^2+157x+93}{(5x+4)(x+1)(2x+6)}$

Exercise B
Simplify the following expression

1) $\dfrac{5}{a+4} - \dfrac{2}{a-2}$
2) $\dfrac{2}{y+1} + \dfrac{3}{y+2}$
3) $\dfrac{3x}{x-1} - \dfrac{x}{x+2}$
4) $\dfrac{1}{n-6} + \dfrac{1}{n-4} - \dfrac{2}{n-5}$
5) $\dfrac{3}{5x} - \dfrac{2}{3x+2} + \dfrac{1}{x+2}$
6) $\dfrac{x+2}{x^2-9} - \dfrac{3}{x-3} + \dfrac{3}{x-3}$
7) $\dfrac{1}{x+2} - \dfrac{2}{x-3} + \dfrac{5}{x}$
8) $\dfrac{3}{2(x^2-1)} - \dfrac{1}{3x+2} + \dfrac{1}{x-1}$

Decomposition of single (Rational) fraction

The break (or split) up of a single fraction into two or more fraction is called decomposition into partial fractions

Denominators with a factorizable quadratic factor

Example 1.5
Express $\dfrac{3x-1}{(x+1)(x-3)}$ in partial fractions

Solution

$$\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$
$$\frac{3x-1}{(x+1)(x-3)} = \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$$

comparing, we have that

$$3x-1 = A(x-3)+B(x+1)$$

we have to methods to find the value of A and B

Method I(Substitution)
we set x to a value that will make one of the terms to vanish; set $x=3$, we have that

$$\begin{aligned} 3(3)-1 &= A(3-3)+B(3+1) \\ 8 &= 0+4B \\ B &= 2 \end{aligned}$$

Again set $x=-1$ we have that

$$\begin{aligned} 3(-1)-1 &= A(-1-3)+B(-1+1) \\ -4 &= -4A+0 \\ A &= 1 \end{aligned}$$

Method II (Comparison)

$$\begin{aligned} 3x-1 &= A(x-3)+B(x+1) \\ &= Ax-3A+Bx+B \\ &= (A+B)x-3A+B \end{aligned}$$

By principle of undetermined coefficient

$$A+B=3 \qquad ; \qquad -3A+B=-1$$

Solving simultaneously we will have that

$$A=1 \qquad ; \qquad B=2$$

Therefore $\dfrac{3x-1}{(x+1)(x-3)} = \dfrac{1}{x+1} + \dfrac{2}{x-3}$

Example 1.6
Write $\dfrac{x-1}{x^2+3x+2}$ in partial fraction

Solution
We first factorize the denominator

$$x^2+3x+2 = (x+2)(x+1)$$

we then decompose $\dfrac{x-1}{(x+2)(x+1)}$

$$\frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$
$$\frac{x-1}{(x+2)(x+1)} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

comparing we have that

$$x-1 = A(x+1)+B(x+2)$$

by method of comparison we let $x=-1$ then we have that

$$\begin{aligned} (-1)-1 &= A(-1+1)+B(-1+2) \\ -2 &= 0+B \\ B &= -2 \end{aligned}$$

In the same way set $x=-2$ we have

$$\begin{aligned} -2-1 &= A(-2+1)+B(-2+2) \\ -3 &= -A+0 \\ A &= 3 \end{aligned}$$

$$\therefore \qquad \frac{x-1}{(x+2)(x+1)} = \frac{3}{x+2} - \frac{2}{x+1}$$

Exercise C
Express the following in partial fractions

1) $\dfrac{x-9}{(x+2)(x-3)}$ 2) $\dfrac{x-3}{x(x-1)(x+1)}$

3) $\dfrac{3x+1}{(x+1)(x-5)}$ 4) $\dfrac{3a-18}{(a+4)(a-2)}$

5) $\dfrac{5y+7}{(y+1)(y+2)}$ 6) $\dfrac{-4}{(x+2)(x+3)}$

7) $\dfrac{2}{(n-6)(n-4)(n-5)}$

Denominator with a non factorizable quadratic factor

In some algebraic fractions, the denominator may contain a factor which is quadratic but not factorizable

Example 1.7
Express $\dfrac{2x+3}{(x-2)(x^2+1)}$ in partial fraction

Solution
Note that x^2+1 is a qudratic expression but not factorizable. Hence to break the expresion into partial fraction, we have that

$$\begin{aligned} \frac{2x+3}{(x+2)(x^2+1)} &= \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \\ &= \frac{A(x^2+1)+(Bx+C)(x+2)}{(x+2)(x^2+1)} \end{aligned}$$

comparing we have that

$$2x+3 = A(x^2+1)+(Bx+C)(x+2)$$

by substitution, set $x = -2$ we have that

$$\begin{aligned}2(-2)+3 &= A((-2)^2+1)+(B(-2)+C)(-2+2)\\ -1 &= 5A+0\\ A &= -\frac{1}{5}\end{aligned}$$

we wont be able to find B and C using this method hence we use the method of comparison; remember

$$\begin{aligned}2x+3 &= A(x^2+1)+(Bx+C)(x+2)\\ &= Ax^2+A+Bx^2+2B+Cx+2C\\ &= (A+B)x^2+(2B+C)x+A+2C\\ 2x+3 &= (A+B)x^2+(2B+C)x+A+2C\end{aligned}$$

By principle of undetermined coefficient we have that

$$2B+C=2 \quad ; \quad A+B=0 \quad ; \quad A+2C=3$$

However we got that $A = -\frac{1}{5}$
Hence

$$A+B=0 \Longrightarrow -\frac{1}{5}+B=0 \Longrightarrow B=\frac{1}{5}$$

Also

$$\begin{aligned}A+2C &= 3\\ -\frac{1}{5}+2C &= 3\\ 2C &= 3+\frac{1}{5}\\ 2C &= \frac{16}{5}\\ C &= \frac{8}{5}\end{aligned}$$

$$\therefore \quad \frac{2x+3}{(x+2)(x^2+1)} = \frac{-\frac{1}{5}}{x+2}+\frac{\frac{1}{5}x+\frac{8}{5}}{x^2+1}$$

Exercise D
Express the following algeraic fractions in their partial fractions

1) $\dfrac{3x-1}{2x(x^2+2)}$ 2) $\dfrac{2x+1}{(x-1)(x^2+1)}$

3) $\dfrac{3x+2}{(x-2)(x^2+3x+1)}$ 4) $\dfrac{3x^2-2x+3}{(x+2)(x^2+x+1)}$

5) $\dfrac{3x}{(2x-1)(x^2+1)}$

Expression with repeated factor in the denominator

In solving problem that contain repeated factor in the denominator, the following should be noted

* To every repeated linear factor in the denominator, assign repeated constant in the numerator

For example, the repeated factor $(x+a)^2$ must be assigned factors such as

$$\frac{A}{x+a}+\frac{B}{(x+a)^2}$$

Example 1.8
Express $\dfrac{2x+5}{(x+4)^2}$ in partial fraction

Solution

$$\begin{aligned}\frac{2x+5}{(x+4)^2} &= \frac{A}{x+4}+\frac{B}{(x+4)^2}\\ \frac{2x+5}{(x+4)^2} &= \frac{A(x+4)+B}{(x+4)^2}\end{aligned}$$

Comparing we have that

$$2x+5 = A(x+4)+B$$

by method of substitution set $x=-4$ we have that

$$\begin{aligned}2(-4)+5 &= A(-4+4)+B\\ -3 &= 0+B\\ B &= -3\end{aligned}$$

set $x=0$ we have that

$$\begin{aligned}2(0)+5 &= A(0+4)+B\\ 5 &= 4A+B\\ 5 &= 4A+(-3)\\ 8 &= 4A\\ A &= 2\end{aligned}$$

$$\therefore \quad \frac{2x+5}{(x+4)^2} = \frac{2}{(x+4)}-\frac{3}{(x+4)^2}$$

Example 1.9
Express $\dfrac{2x+5}{(x+2)(x+4)^2}$ in partial fraction

Solution

$$\begin{aligned}\frac{2x+5}{(x-2)(x+4)^2} &= \frac{A}{x-2}+\frac{B}{x+4}+\frac{C}{(x+4)^2}\\ &= \frac{A(x+4)^2+B[(x-2)(x+4)]+C(x-2)}{(x+4)^2(x-2)}\end{aligned}$$

comparing we will have that

$$2x+5 = A(x+4)^2+B[(x-2)(x+4)]+C(x-2)$$

by method of substitution set $x=-4$ we have that

$$\begin{aligned}-8+5 &= 0+0+C(-6)\\ -3 &= -6C\\ C &= \frac{1}{2}\end{aligned}$$

Similarly set $x=2$ we have that

$$\begin{aligned}4+5 &= A6^2+0+0\\ 9 &= 36A\\ A &= \frac{1}{4}\end{aligned}$$

set $x = 0$ we have that

$$\begin{aligned}
5 &= A4^2 + B(-8) + C(-2) \\
5 &= 16A - 8B - 2C \\
5 &= 16\left(\frac{1}{4}\right) - 8B - 2\left(\frac{1}{2}\right) \\
5 &= 4 - 8B - 1 \\
8 &= 8B \\
B &= 1
\end{aligned}$$

$A = \frac{1}{4}$; $B = 1$; $C = \frac{1}{2}$

$$\therefore \quad \frac{2x+5}{(x+4)^2(x-2)} = \frac{\frac{1}{4}}{x-2} + \frac{1}{x+4} + \frac{\frac{1}{2}}{(x+4)^2}$$

Exercise E
Express the following in partial fraction

1) $\dfrac{2x+5}{(x+2)(x+1)^2}$

2) $\dfrac{x+3}{(x+2)^2}$

3) $\dfrac{x+3}{(x-1)^2}$

4) $\dfrac{x+1}{2x^2+x-1}$

5) $\dfrac{x^2-2}{x^2(x+2)}$

6) $\dfrac{5x+2}{(x+1)(x^2-x+1)}$

7) $\dfrac{2x^2-3x+5}{(x-1)(x+3)^2}$

8) $\dfrac{3x^3+5x^2+7}{(x^2+3)(x^2+x-1)}$

9) $\dfrac{x+1}{16-x^2}$

10) $\dfrac{2x^2+10x+8}{(x+2)(2x+3)(3x+5)}$

11) $\dfrac{x}{(x-1)(x+3)}$

12) $\dfrac{5x^3-10x^2+3x+5}{(x^2+x-1)(x^2+1)}$

13) $\dfrac{3x^2+7}{(x^2+3)(x^2-x+1)}$

14) $\dfrac{3x^2+x+2}{x(x^2+7x+1)}$

15) $\dfrac{5x^2+2x+10}{(x+5)(x+1)^3}$

16) $\dfrac{x^3+3x^2+x+5}{(x^2-4)^2}$

WEEK TWO
SEQUENCE

Consider the following set of numbers

1. 2, 4, 6, 8,...
2. 1, 3, 5, 7, ...
3. 1, 4, 9, 16, ...
4. 0, 1, 1, 2, 3, 5, 8, 13,...
5. 1, 3, 6, 10, ...

Can you predict the next three numbers in each set of numbers by identifying the pattern. Each set of numbers above is called a sequence.

A sequence is a set of number in definite order in which members of the set can be predicted.

Each members of a sequence is called a term.

nth term of a Sequence

Sometimes the general form of a sequence is given

Example 2.0
Write down the first six terms of the sequence

$$T_n = 1 - 2n$$

Solution

$$\begin{aligned}
\text{first term} \quad T_1 &= 1 - 2(1) \\
&= -1 \\
\text{2nd term} \quad T_2 &= 1 - 2(2) \\
&= 1 - 4 \\
&= -3 \\
\text{3rd term} \quad T_3 &= 1 - 2(3) \\
&= 1 - 6 \\
&= -5 \\
\text{4th term} \quad T_4 &= 1 - 2(4) \\
&= 1 - 8 \\
&= -7 \\
\text{5th term} \quad T_5 &= 1 - 2(5) \\
&= 1 - 10 \\
&= -9 \\
\text{6th term} \quad T_6 &= 1 - 2(6) \\
&= 1 - 12 \\
&= -13
\end{aligned}$$

Example 2.1
Find the 2nd, 3rd and 5th term of the sequence

$$T_n = (-2)^n$$

Solution

$$\begin{aligned}
\text{2nd term} &= T_2\\
T_2 &= (-2)^2\\
&= -2 \times -2\\
&= 4\\
\text{3rd term} &= T_3\\
T_3 &= (-2)^3\\
&= -2 \times -2 \times -2\\
&= -8\\
\text{5th term} &= T_5\\
T_5 &= (-2)^5\\
&= -32
\end{aligned}$$

Example 2.2
Given the sequence

$$5, 1, -3, ..$$

(a) A formula for the nth term (b) The 30th term (c) Which terms of the sequence are -75 and -200

Solution

$$\begin{aligned}
&T_1 = 5\\
&T_2 = 5 - 4 = 1\\
&T_3 = 5 - 4 - 4 = 5 - 2 \times 4 = -3\\
&T_4 = 5 - 4 - 4 - 4 = 5 - 3 \times 4\\
&T_5 = 5 - 4 \times 4\\
&.\\
&.\\
&.\\
&T_n = 5 - (n-1) \times 4
\end{aligned}$$

$$\begin{aligned}
\text{30th term is } &T_{30}\\
T_{30} &= 9 - 4(30)\\
&= 9 - 120\\
&= -111
\end{aligned}$$

$$\begin{aligned}
T_n &= 75\\
9 - 4n &= 75\\
-4n &= 75 - 9\\
n &= 21
\end{aligned}$$

$$\begin{aligned}
T_n &= -200\\
9 - 4n &= -200\\
-4n &= -209\\
n &= \frac{209}{4} \quad ?
\end{aligned}$$

But n has to be an integer (Whole number). Hence -200 is not a term in the sequence

Exercise F

1. Say what the pattern is for each of the following sequence and if possible, write the general form of the sequence
   (a) 20, 17, 14, ...
   (b) 1, 8, 27, 64,...
   (c) 8, 0.8, 0.08, 0.008, ...
   (d) 3, -6, 12, -24, ...
   (e) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$
   (f) 0, 1, 1, 2, 3, 5, 8, ...
   (g) 2, 5, 10, 17, ...
   (h) 2, 6, 12, 20, ...
2. The nth term of a sequence is given. Write down the first four (4) terms of each sequence
   (a) $3n + 1$ (b) $2n - 1$ (c) n^2
   (d) $\frac{n+1}{2}$ (e) 2^{n-1} (f) $5 - 2n$
   (g) $2n^2 - 1$ (h) $(-1)^{n+1}$ (i) $(-1)^n \times n$
   (j) $\frac{n}{n+1}$ (k) 16×2^{-n} (l) $n(n+1)$
3. T_n is given, for each of the sequence find
   (i) T_4 (ii) T_8 (iii) T_{12}
   (a) $2n + 1$ (b) $3 - 5n$ (c) $\frac{n-1}{n+1}$
   (d) $n^2 - 1$ (e) n (f) $\frac{n}{n+2}$
4. Find a formular in terms of n for T_n in each of the following sequence
   (a) 2, 4, 6, 8, ...
   (b) -7, -4, -1, 2, ...
   (c) 1, 4, 9, 16,...
   (d) 1, 2, 4, 8, 16
5. If $T_n = 2 - 3n$, which term of the sequence is
   (a) -58 (b) -100 (c) -110

Types of sequence

We would be looking at two type of sequence viz

- Arithmetic progression
- Geometric Progression

Arithmetic Progression (AP)

The Arithemetic progression (sometimes referred as Linear Sequence) is a type of sequence in which the difference between a term and the term immediately preceding it is always the same (a constant). This constant number is called the common difference.

Examples are

1. 2, 4, 6, 8, 10, ...
2. 1, 3, 5, 7, 9, 11, ...
3. -1, -2, -3, -4, -5, ...
4. 5, 10, 15, 20, 25, ...

The nth term of an AP

Let a be the first term of an AP and d the common difference then

$$\begin{aligned} T_1 &= a \\ T_2 &= a + d \\ T_3 &= a + d + d \\ &= a + 2d \\ T_4 &= a + 2d + d \\ &= a + 3d \\ T_5 &= a + 4d \\ &. \\ &. \\ &. \\ T_n &= a + (n-1)d \end{aligned}$$

Hence the nth term of a sequence is given as

$$T_n = a + (n-1)d$$

Example 2.3
Find the 40th term of the AP 6, 11, 16, 21, ...

Solution

$$\begin{aligned} &\text{40th term is } T_{40} \\ &T_{40} = a + 3d \\ &\quad\;\; = 6 + 39(5) \\ &\quad\;\; = 201 \end{aligned}$$

The 40th term is 201

Example 2.4
What is the 15th term of the sequence -3, 2, 7, ...

Solution
This sequence is an AP

$$\begin{aligned} T_{15} &= a + 14d \\ &= -3 + 14(5) \\ &= -3 + 70 \\ T_{15} &= 69 \end{aligned}$$

Example 2.5
The 4th term of an AP is 15 and the 9th term is 35. Find the 15th term.

Solution

$$T_4 = 15 \quad ; T_9 = 35$$

But $T_4 = a + 3d$; $T_9 = a + 8d$ hence

$$a + 3d = 15 \qquad (1)$$

$$a + 8d = 35 \qquad (2)$$

the above is a simultaneous equation, using elimination method we have that

$$5d = 20 \quad \Leftrightarrow \quad d = 4$$

Substituting $d = 4$ in (1) we have

$$\begin{aligned} a + 3(4) &= 15 \\ a + 12 &= 15 \\ a &= 3 \end{aligned}$$

To find T_{15} we have that

$$\begin{aligned} T_{15} &= a + 14d \\ &= 3 + 14(4) \\ &= 3 + 56 \\ T_{15} &= 59 \end{aligned}$$

Example 2.6
Find the formular for the nth term of the AP 15, 20, 25, 30, ...

Solution

$$\begin{aligned} T_n &= a + (n-1)d \\ T_n &= 15 + (n-1)5 \\ &= 15 + 5n - 5 \\ T_n &= 10 + 5n \end{aligned}$$

Exercise G

1. Find the 25th term of the AP whose first term is 8 and common difference is 2
2. Find the 20th term of the AP 12, 9, 6, 3,...
3. If -3, x, y, 12 is an AP find x and y
4. Find the value of n given that 66 is the nth term of the AP 1, 6, 11, 16, ...
5. The fourth term of an AP is 14 and the 12th term is 70, find the first term and common difference
6. Given an AP 3, 7, 11, ... find (a) the 8th, 15th, 50th term (b) a formular for the nth term of the AP
7. The 5th and 10th term of an AP are -12 and -27 respectively. Find its 15th term
8. If the first and last term of an AP are $2\frac{1}{2}$ and 19 respectively, how many terms has the AP if the common difference is $1\frac{1}{2}$
9. The sum of the 3rd and 7th terms of an AP is 38, and the 9th term is 37. Find the first four terms of the AP if the 10th term of an AP is double the 2nd term, find the 8th term given that its first term is 7

10. A man receives an annual increase of 30 NGN and after a period of 10 years service the man has a salary of 1014 NGN. Find his initial salary.
11. The first term of an AP is 3 and the eleventh term is 18. Find the number of terms in the progression if the sum is 81
12. What is the 100th term of the sequence 2, 5, 8, 11, 14, ...

Arithmetic means

If a, b, c are three consecutive terms of an AP, then b is the arithmetic mean of a and c.

Notice that since

$$a, b, c, ... \qquad \text{is an AP}$$

then

$$b - a = c - b \qquad \text{same common difference}$$

$$\begin{aligned} b - a &= c - b \\ b + b &= c + a \\ 2b &= c + a \\ b &= \frac{c + a}{2} \end{aligned}$$

therefore b is the average of a and c

Example 2.7
Find the arithmetic mean of 4 and 18

Solution
Let x be the arithmetic mean then

$$4,\ x,\ 18, ... \qquad \text{is an AP}$$

$$\begin{aligned} x &= \frac{4 + 18}{2} \\ x &= 11 \end{aligned}$$

Example 2.8
Insert three arithmetic mean between 19 and 35

Solution
let the arithmetic mean be x, y, z then

$$19,\ x,\ y,\ z,\ 35,\ ...$$

is an AP hence

$$\begin{aligned} a &= 19 \\ T_5 &= 35 \end{aligned}$$

But

$$\begin{aligned} T_5 &= a + 4d \\ 35 &= 19 + 4d \\ 35 - 19 &= 4d \\ 16 &= 4d \\ d &= 4 \end{aligned}$$

$a = 19$; $d = 4$; we go on to find x, y and z

$$\begin{aligned} x &= T_2 \\ x &= a + d \\ x &= 19 + 4 \\ x &= 23 \end{aligned}$$

$$\begin{aligned} y &= T_3 \\ y &= a + 2d \\ y &= 19 + 8 \\ y &= 27 \end{aligned}$$

$$\begin{aligned} z &= T_4 \\ z &= a + 3d \\ z &= 19 + 12 \\ z &= 31 \end{aligned}$$

Exercise Gi

1. Find the arithmetic mean of
   (a) -10 and -5 (b) 7 and 3
   (c) -8 and 0 (d) $-2\frac{1}{2}$ and $5\frac{1}{2}$
2. Insert 4 arithmetic mean between -12 and 13
3. Find 3 arithmetic mean between a and b

Geometric progression

A geometric progression(refered sometimes as an exponential sequence) is atype of sequence in which the ratio of any of its two consecutive terms are always the same (constant). This constant number is called the common ratio.

In other words if a sequence

$$T_n = T_1,\ T_2,\ T_3,\ T_4,\ ...$$

is such that

$$\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3}, ...$$

then T_n is a geometric progression

Exercise H
Find the common ratio in each of the following GP

1. 6, 12, 24, 48, ...
2. 36, 12, 4, $\frac{1}{3}$, ...
3. 4, -8, 16, -32, ...
4. 54, -18, 6,

The nth term of a GP

Let a be the first term of a GP and r the common ratio, then

$$\begin{aligned}
T_1 &= a \\
T_2 &= a \times r = ar \\
T_3 &= ar \times r = ar^2 \\
T_4 &= ar^3 \\
T_5 &= ar^4 \\
&. \\
&. \\
&. \\
T_n &= ar^{n-1}
\end{aligned}$$

Hence the nth term of a GP is

$$T_n = ar^{n-1}$$

Example 2.9
Find the 9th term of the sequence 2, -10, 50, ...

Solution
The sequence is GP given that

$$-\frac{10}{2} = -\frac{50}{10} = -5$$

$$\begin{aligned}
T_9 &= ar^{9-1} \\
T_9 &= ar^8
\end{aligned}$$

but $a = 2$; $r = -5$

$$T_9 = 2 \times (-5)$$

Example 2.91
A sequence is given as 128, 64, 32, ... (a) Find the 12th term of the sequence (b) a formula for the nth term

Solution
The Sequence is GP since

$$\frac{64}{128} = \frac{32}{64} = \frac{1}{2}$$

$$\begin{aligned}
T_{12} &= ar^{11} \\
&= 128 \times \left(\frac{1}{2}\right)^{11} \\
&= \frac{128}{2^{11}} \\
&= \frac{2^7}{2^{11}} \\
&= \frac{1}{2^4}
\end{aligned}$$

$$T_{12} = \frac{1}{2^4}$$

$$\begin{aligned}
T_n &= ar^{n-1} \\
&= 128 \times \left(\frac{1}{2}\right)^{n-1} \\
&= \frac{2^7}{2^{n-1}} \\
&= 2^{7-(n-1)} \\
&= 2^{8-n}
\end{aligned}$$

$$T_n = 2^{8-n}$$

Example 2.92
The second term of an exponential sequence is 35 and the fourth term is 875. find (a) the first term (b) the fifth term

Solution

$T_2 = 35$; $T_4 = 875$;

but $T_2 = ar = 35$; $T_4 = ar^3 = 875$

$$\begin{aligned}
\frac{ar}{ar^3} &= \frac{35}{875} \\
\frac{1}{r^2} &= \frac{1}{25} \\
r^2 &= 25 \\
r &= \sqrt{25} \\
r &= 5
\end{aligned}$$

(a) we now find a

$$\begin{aligned}
ar &= 35 \\
5a &= 35 \\
a &= 7
\end{aligned}$$

(b)

$$\begin{aligned}
T_5 &= ar^4 \\
&= 7 \times 5^4 \\
&= 7 \times 625 \\
&= 4375
\end{aligned}$$

Example 2.93
A GP is such that the third term minus the first term is 48. The fourth term minus the second term is 144. Find (a) The common ratio (b) the first term (c) the sixth term of the sequence

Solution

$$T_3 - T_1 = 48$$

$$T_4 - T_2 = 144$$

$T_3 = ar^2$, $T_1 = a$, $T_4 = ar^3$, $T_2 = ar$

$$ar^2 - a = 48 \qquad (1)$$
$$ar^3 - ar = 144 \qquad (2)$$

Multiply (1) by r we have

$$ar^3 - ar = 48r \qquad (3)$$

Considering (2) and (3) and by method of elimination

$$\begin{aligned} 144 - 48r &= 0 \\ 48r &= 144 \\ r &= \frac{144}{48} \\ r &= 3 \end{aligned}$$

Substituting $r = 3$ in (1) we have

$$\begin{aligned} a(3)^2 - a &= 48 \\ 9a - a &= 48 \\ 8a &= 48 \\ a &= 4 \end{aligned}$$

$$\begin{aligned} \text{sixth term} &= T_6 \\ T_6 &= ar^5 \\ T_6 &= 4 \times 3^5 \\ &= 4 \times 243 \\ &= 1458 \end{aligned}$$

$$T_6 = 1458$$

Exercise I

1. Write down the first 4 terms of the GP whose first term is 48 and common ratio is $\dfrac{1}{2}$
2. Find the common ratio of a GP whose first term is 2 and fourth term is 54.
3. What is the 16th term of the GP 16, -8, 4,...
4. What is the nth term of the GP 1, $2x$, $4x^2$,...
5. State a formular in terms of n for T_n in its simplest form for each of the following GPs
   (a) 1, 2, 4, ...
   (b) 15, 5, $1\frac{2}{3}$, ..
   (c) 80, 20, 5, ...
   (d) 50, 20, 8, ...
   (e) 1, $\sqrt{2}$, 2, ...
   (f) 1, p^2, p^4, ...
6. The 2nd and 6th term of a GP ($r > 0$) are $\frac{8}{9}$ and $4\frac{1}{2}$ respectively. Find a and r
7. The 1st and 7th terms of a G.P. are $40\frac{1}{2}$ and $\frac{1}{18}$ respectively. Find the 2nd term.
8. The 3rd and 7th terms of a GP are 81 and 16 respectively. Find the 1st and 5th terms
9. The sum of the first and 3rd terms of a GP is $2\frac{1}{3}$ and the sum of the 2nd and 4th terms is $1\frac{3}{4}$. Find a and r
10. When a ball is dropped onto the floor, it always rebounds a distance equal to $\frac{2}{3}$ of the height it fell. If the original height was h m, how high will it rise after the 5th bounce

Geometric Mean

If a, b, c are consecutive terms of a GP, then b is the geometric mean of a and c

Note that since

$$a,\ b,\ c,$$

... is a GP. Then

$$\frac{b}{a} = \frac{c}{b}$$

$$\Rightarrow \quad b^2 = ac$$

$$\Rightarrow \quad b = \sqrt{ac}$$

Example 2.94
State the geometric mean of $1\frac{1}{2}$ and $\frac{8}{27}$

Solution
Let b be the geometric mean, then

$$1\tfrac{1}{2},\ b,\ \tfrac{8}{27},\ ...$$

is a GP

$$\begin{aligned} b &= \sqrt{1\frac{1}{2} \times \frac{8}{27}} \\ &= \sqrt{\frac{3}{2} \times \frac{8}{27}} \\ &= \sqrt{\frac{4}{9}} \\ b &= \frac{2}{3} \end{aligned}$$

Example 2.95
Find three geometric mean between $\frac{4}{27}$ and 12

Solution
If the geometrc means are x, y, z, then

$$\tfrac{4}{27},\ x,\ y,\ z,\ 12,...$$

is a GP $\Rightarrow \quad a = \frac{4}{27}$ and $T_5 = 12$
But $T_5 = ar^4$

$$\begin{aligned} ar^4 &= 12 \\ \frac{4}{27} \times r^4 &= 12 \\ r^4 &= \frac{12 \times 27}{4} \\ r^4 &= 81 \\ r &= 81^{\frac{1}{4}} \\ r &= 3 \end{aligned}$$

since a and r have been determined we now find x, y and z

$$\begin{aligned} x &= T_2 \\ x &= arx \qquad = \frac{4}{27} \times 3 \\ x &= \frac{4}{9} \end{aligned}$$

$$y = T_3$$
$$y = ar^2$$
$$y = \frac{4}{27} \times 3^2$$
$$y = \frac{4}{3}$$

$$z = T_4$$
$$z = ar^3$$
$$z = \frac{4}{27} \times 3^3$$
$$z = 4$$

Exercise J

1. State the geometric mean of

   (a) 5 and 20 (b) $\frac{1}{2}$ and 16 (c) $1\frac{1}{2}$ and $\frac{8}{27}$

2. The arithmetic mean of two numbers is 15, and the geometric mean is 9. Find the numbers.

3. Find four geometric means between $\frac{1}{8}$ and 4

4. If 8, x, y, 27 is a GP. Find x and y

WEEK THREE
SERIES

Consider the sequence T_1, T_2, T_3, ... summing the terms we have that

Sum to the first term

$$T_1 = S_1$$

Sum to the 2nd term

$$T_1 + T_2 = S_2$$

Sum to the 3rd term

$$T_1 + T_2 + T_3 = S_3$$

Sum to the nth term

$$T_1 + T_2 + T_3 + ... + T_n = S_n$$

S_n is called a series of n terms. A series is a partial sum of a sequence

Infact if

2, 4, 6, ...

is a sequence. Then

2+4+6+...

is a series

Sum of an AP

$$\begin{aligned}
S_n &= T_1 + T_2 + T_3 + ... + T_n \\
S_n &= [a] + [a + d] + [a + 2d] + ... + [a + (n-1)d] \\
S_n &= [a + (n-1)d] + [a + (n-2)d] + a + (n-3)d] + ... + [a] \\
2S_n &= n[2a + (n-1)d] \\
S_n &= \frac{n}{2}[2a + (n-1)d]
\end{aligned}$$

Hence the sum of first n terms of an AP is given as

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Example 3.1
Find the sum of the first 20 terms of

$$2 + 5 + 8 + ...$$

Solution

$$\begin{aligned}
S_{20} &= \frac{20}{2}\,[2(2) + (20-1) \times 3] \\
&= 10 \times 61 \\
&= 610
\end{aligned}$$

Example 3.2
Find the sum of the first 10 terms of the linear sequence 1, 5, 9, 13, ...

Solution

$$\begin{aligned}S_n &= \frac{n}{2}\left[2a+(n-1)d\right]\\ S_{10} &= \frac{10}{2}\left[2a+9d\right]\\ S_{10} &= 5[2(1)+9(4)]\\ &= 5[2+36]\\ &= 5(38)\\ S_n &= 190\end{aligned}$$

Example 3.3
The sum of 8 terms of an AP is 12 and the sum of 16 terms is 56. Find the AP

Solution

$$S_8 = 12 \quad ; \quad S_{16} = 56$$

$$S_8 = \frac{8}{2}\left[2a+7d\right] \quad ; \quad S_{16} = \frac{16}{2}\left[2a+15d\right]$$

$$\begin{aligned}12 &= \frac{8}{2}\left[2a+7d\right]\\ 12 &= 4[2a+7d]\\ 3 &= 2a+7d \qquad (1)\\ S_{16} = 56 &= \frac{16}{2}\left[2a+15d\right]\\ 56 &= 8[2a+15d]\\ 7 &= 2a+15d\end{aligned}$$

Solving simultaneously we have that

$$a = -\frac{1}{4} \quad \text{and} \quad d = \frac{1}{2}$$

Exercise K

1. Find the sum of 12 terms of the AP -7, -3, 1, ... starting from the 8th term
2. Find the sum of the multiples of 7 between 50 and 200
3. Find the sum stated for each AP
   (a) 20 terms of -9, -4, 1, ...
   (b) 100 terms of 1, 3, 5, ...
   (c) 40 terms of 8, 7.5, 7, ...
   (d) 25 terms of $5\frac{1}{2}$, 4, $2\frac{1}{2}$, ...
   (e) 9 terms of -23, -17, -11, ...
4. The sum of the first 10 terms of an AP is 15 and the sum of the next 10 terms is 215. find the AP.
5. For an AP, $a = 25$, $d = -3$. Find the value of n if $S_n = 112$
6. Find the sum of all the multiples of 3 between 100 and 301. Hence find the sum of all the numbers between 100 and 301 inclusive, which are not multiples of 3
7. The fourth term of an AP is 8 and the sum of the first 12 terms is 126. Find the AP
8. The sum of the first n terms of an AP is $2n$, and the sum of the first $2n$ terms is $3n$. Find the sum of the first $3n$ terms

Sum of the first nth term of a GP

$$\begin{aligned}S_n &= a+ar+ar^2+...+ar^{n-1}\\ rS_n &= ar+ar^2+ar^3+...+ar^n\\ S_n - rS_n &= a-ar+ar-ar^2+...-ar^{n-1}+ar^{n-1}-ar^n\\ S_n - rS_n &= a-ar^n\\ S_n(1-r) &= a(1-r^n)\\ S_n &= \frac{a(1-r^n)}{1-r}\end{aligned}$$

Hence the sum of the first n terms of a Geometric progression

$$S_n = \frac{a(1-r^n)}{1-r}$$

Example 3.4
Find the sum of the first 10 terms of

$$\tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{2}, ...$$

Solution

$$\begin{aligned}S_n &= \frac{a(1-r^n)}{1-r}\\ S_{10} &= \frac{\frac{1}{8}\left(1-2^{10}\right)}{1-2}\\ &= \frac{\frac{1}{8}(1-1024)}{-1}\\ &= \frac{1023}{8}\\ S_{10} &= 127\frac{7}{8}\end{aligned}$$

Sum to infinity

The sum of the n terms as n approaches to infinity is called the sum to infinity of the series and is designated as S_∞ and

$$S_\infty = \frac{a}{1-r}$$

Exercise L

1. Find the sum of 6 terms of the GP 32, 16, 8, ...
2. Find the sum of 17 terms of the GP 5, 1, $\frac{1}{5}$, ...
3. The 2nd and 4th terms of a GP are 21 and 189. Find the sum of the first 7 terms
4. The sum of the first two terms of a GP is $2\frac{1}{2}$ and the sum of the first four terms is $3\frac{11}{18}$. Find the GP. ($r.0$)
5. The 2nd and 6th terms of a GP are $\frac{8}{9}$ and $4\frac{1}{2}$ respectively. Find the sum of the first 6 terms.

6. A ball is thrown vertically upwards a distance of 81 cm from a floor. After each bounce on the floor, the ball rises to a height of $\frac{2}{3}$ of the distance it previously fell. Show that the total distance travelled by the ball, until it reaches the floor for the nth bounce is given by

$$486\left[1-\left(\frac{2}{3}\right)^n\right]$$

7. In a GP, the product of the 2nd and 4th terms is double the 5th term, and the sum of the first 4 terms is 80. Find the GP

WEEK FOUR
INEQUALITIES

Linear inequalities in one variable

We introduce new symbols used to relate the inequality of any two numbers.

$>$ means "greater than". Hence "6 is greater than 4" can be written as

$$6 > 4$$

"$<$" means "less than". Hence "3 is less than 20" can be written as

$$3 < 20$$

Also
"$\geq$" means "greater than or equal to"
"$\leq$" means "less than or equal to "

Basic rules of inequalities

1. If $a > b$ then
$$a + x > b + x$$
$$a - x > b - x$$
Any quantity can be added or subtracted on both sides of an inequality.

   For example
$$5 > 2$$
then
$$5 + 2 > 2 + 2$$
The same rule works when $a < b$

2. If $a > b$ then
$$ax > bx$$
$$\frac{a}{x} > \frac{b}{x}$$
Both sides of an inequality can be multiplied or divided by a positive quantity without changing the inequality sign. For example
$$12 > 4$$
$$12(2) > 4(2) \Rightarrow 24 > 8$$
also
$$\frac{12}{4} > \frac{4}{2} \Rightarrow 3 > 2$$
The same rule works when $a < b$

3. If $a > b$ and $-x$ is negative then
$$a(-x) < b(-x)$$
also
$$\frac{a}{-x} < \frac{a}{-x}$$
If both sides of an inequality are multiplied or divided by a negative quantity, the inequality sin must be reversed.

   The same rule works when $a < b$

Graphical Interpretation of Linear Inequalities in one variable

The solution to a linear inequality constitutes a set of numbers. For instance, when you say

$$x = 3$$

it means that $x4$ is a single number which is 3. But when you say

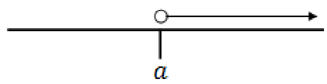
$$x \geq 3$$

it means that x can be number from 3 and above say

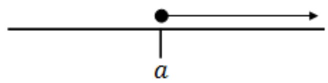
$$3, 4, 5, 6, \ldots$$

Linear Inequalities in one variable can be represented on a number line.

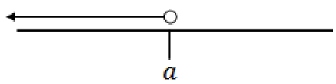
For $x > a$



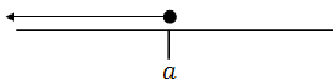
For $x \geq a$



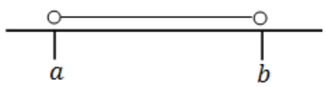
For $x < a$



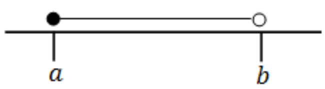
For $x \leq a$



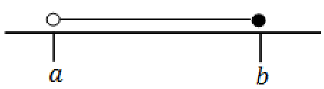
For $a < x < b$



For $a \leq x < b$



For $a < x \leq b$



For $a \leq x \leq b$

Exercise M

Represent each of the following inequalities graphically

(a) $x > 3$ (b) $x < -2$

(c) $x \geq 4$ (d) $x \leq -1$

(e) $-2 < x < 3$ (f) $-1 \leq x \leq 4$

(g) $-3 < x \leq 2$ (h) $-2 \leq x < 1$

Solving Linear inequalities in one variable

Linear inequalities in one variable are solved in the same way as linear equation but note that if multiplying/dividing both sides by a negative number, the inequality sign have to be reversed.

Example 4.1

Find the solution set of

$$2x - 3 < x + 7$$

and represent the solution on a number line

Solution

$$\begin{aligned} 2x - 3 &< x + 7 \\ 2x + x &< 7 + 3 \\ 3x &< 10 \\ x &< \frac{10}{3} \end{aligned}$$

Example 4.2

Solve $\frac{x}{3} - 2 \leq \frac{2}{5}x + \frac{1}{4}$

Solution

$$\begin{aligned} \frac{x}{3} - 2 &\leq \frac{2x}{5} + \frac{1}{4} \\ \frac{x}{3} - \frac{2x}{3} &\leq \frac{1}{4} + 2 \\ \frac{5x - 6x}{15} &\leq \frac{1+8}{4} \\ -\frac{x}{15} &\leq \frac{9}{4} \\ -4x &\leq 135 \\ x &\geq -\frac{135}{4} \end{aligned}$$

Example 4.3

Given the inequality

$$\frac{x+1}{2} + \frac{2x-1}{3} \geq 1$$

Show the interval in which x can lie if it satisfy the above inequality

Solution

$$
\begin{aligned}
\frac{x+1}{2}+\frac{2x-1}{3} &\geq 1\\
\frac{3(x+1)+2(x-1)}{6} &\geq 1\\
3x+3+2x-2 &\geq 6\\
5x+1 &\geq 6\\
5x &\geq 5\\
x &\geq 1
\end{aligned}
$$

Example 4.4
Solve the inequality below, showing the interval(s) obtained on a number line

$$-5 \leq \frac{x}{3}-1 \leq 1$$

Solution

First we disintegrate the combined inequality into

$$-5 \leq \frac{x}{3}-1 \quad ; \quad \frac{x}{3}-1 \leq 1$$

we then solve separately

$$
\begin{aligned}
-5 &\leq \frac{x}{3}-1\\
-4 &\leq \frac{x}{3}\\
-12 &\leq x
\end{aligned}
$$

$$
\begin{aligned}
\frac{x}{3}-1 &\leq 1\\
\frac{x}{3} &\leq 2\\
x &\leq 6
\end{aligned}
$$

Combining $-12 \leq x$ and $x \leq 6$ we have

$$-12 \leq x \leq 6$$

Exercise M

1. Find the solution set of the inequality
$$\frac{1}{2}x+3 > 1-x$$

2. Find the range of values of x for which the inequality
$$4x+3 \geq 2x-7$$
is true

3. Solve and represent the solution set of each of the following inequality graphically

   (a) $2x+1 \leq x+5$ (b) $\frac{1}{3}x+2 \geq \frac{1}{2}x+1$

   (c) $3x-2 \geq \frac{1}{2}x+3$ (d) $4x+1 \leq 3x+2$

   (e) $2-3x \geq x-3$ (f) $x+\frac{1}{2} \geq 2x+5$

4. Show graphically the solution set of

   (a) $3x-4 < 3x+1 \leq 3x+2$
   (b) $2x+4 < x+3 < 4x-5$
   (c) $1 \leq \frac{3x+1}{4} < 2$
   (d) $-\frac{1}{2} \leq \frac{3-2x}{5} \leq 1$

5. Find the interval(s) in which x can lie if

   (a) $3x-1 > 5$ or $1-2x \geq 4$
   (b) $3x-1 > 5$ and $1-2x \geq 4$
   (c) $\frac{x-1}{2} \geq \frac{2}{3}$ and $\frac{2x-1}{3} > 1$

6. If $2 \leq x \leq 3$ and $1 \leq y \leq 12$, find the greatest and least values of

   (a) $x+y$ (b) $x-y$

   (c) $\frac{1}{x}+\frac{1}{y}$ (d) x^2-y^2

7. Consider the following statements about real numbers. Decide if they are always true or not. If they are not true, make up a numerical example to show why

   (a) if $ax > bx$ then $a > b$
   (b) if $a > b$ and $b > c$, then $a > c$
   (c) if $a > b$ and $b < c$, then $a < c$
   (d) if $a > b$ and $c > d$, then $a+c > b+d$
   (e) if $a > b$ and $c > d$, then $a-c > b-d$
   (f) if $a > b$ and $c > d$, then $ac > bd$
   (g) if $a^2 > b^2$, then $a > b$

WEEK FIVE
LINEAR INEQUALITY IN TWO VARIABLE

The solution set of linear inequality in two variable can only be properly determined by representing it graphically on an x - y plane

Example 4.1
Show graphically the region represented by the inequality

$$2x + y + 1 > 0$$

Solution

$$2x + y + 1 > 0$$
$$y > -2x - 1$$

First, we sketch the line

$$y = -2x - 1$$

The required region will be on one of the two sides of line

(***To be Sketched in class***

Example 4.2
Show graphically the regions represented by the inequality

$$2x - y \geq 4 + y - 2x$$

Solution
We first simplify

$$2x - y \geq 4 + y - 2x$$
$$2x - y + 2x - y \geq 4$$
$$4x - 2y \geq 42x - y \qquad \geq 2$$

First we sketch the line

$$2x - y = 2$$

The required region will be on one of the line

(***To be Sketched in class***

Exercise N
On a graph paper, show the half-plane with full or dotted boundary lines as necessary for the following inequalities

(a) $y > 2x$ (b) $y < x$

(c) $y \leq x - 2$ (d) $x + y \leq 4$

(e) $x - 2y \leq 2$ (f) $2x - y < 4$

(g) $y \geq 3 - 2x$ (h) $4x + 3y \geq 12$