WEEK ONE
INDICES

Indices are numbers in the form

$$x^a$$

where x is called the BASE and a is called the index

Converting numbers to their indices form
Examples are

$4 = 2^2$ $\quad$ $12 = 3 \times 2^2$
$9 = 3^2$ $\quad$ $24 = 3 \times 2^3$
$8 = 2^3$ $\quad$ $21 = 3^1 \times 7^1$

LAWS OF INDICES

1. $x^a \times x^b = x^{a+b}$
2. $x^a \div x^b = \dfrac{x^a}{x^b} = x^{a-b}$
3. $x^0 = 1$
4. $x^{-a} = \dfrac{1}{x^a}$

Example 1.1
Simplify $10^5 \times 10^4$
Solution

$$\begin{aligned} 10^5 \times 10^4 &= 10^{5+4} \\ &= 10^9 \end{aligned}$$

Example 1.2
Simplify $x^3 \div x^{-5}$
Solution

$$\begin{aligned} x^3 \div x^{-5} &= x^{3-(-5)} \\ &= x^8 \end{aligned}$$

Example 1.3
Simplify $5y \times 4y^4$
Solution

$$\begin{aligned} 5y \times 4y^4 &= 5 \times 4 \times y \times y^4 \\ &= 20 \times y^{1+4} \\ &= 20 \times y^{20} \\ &= 20y^5 \end{aligned}$$

Example 1.4
Simplify $\dfrac{24x^6}{8x^4}$
Solution

$$\begin{aligned} \frac{24x^6}{8x^4} &= \frac{24}{8} \times \frac{x^6}{x^4} \\ &= 3 \times x^{6-4} \\ &= 3 \times x^2 \\ &= 3x^2 \end{aligned}$$

Exercises A
Simplify

1. $m^8 \div m^5$ $\quad$ 2. $c^7 \div c$ $\quad$ 3. 2^0
4. $6 \times z^0$ $\quad$ 5. 4^{-3} $\quad$ 6. $3x^{-3}$
7. $\left(\dfrac{1}{4}\right)^{-2}$ $\quad$ 8. $\left(\dfrac{2}{3}\right)^{-1}$ $\quad$ 9. $x^3 \div x^{-5}$
10. $a^{-9} \div b^0$ $\quad$ 11. $9a^{-5} \times 4a^6$ $\quad$ 12. $5x^2 \times 4x^0 \times 2x^{-6}$
11. $\dfrac{15 \times 10^4}{3 \times 10^{-2}}$

WEEK TWO
INDICES (continue)

LAW OF INDICES

5. $(x^a)^b = x^{ab}$
6. $x^{\frac{a}{b}} = (\sqrt[b]{x})^a = \sqrt[b]{x^a}$
7. $\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$
8. $(xy)^a = x^a y^a$

Example 2.1
Simplify $(a^3)^2$
Solution

$$\begin{aligned} (a^3)^2 &= a^{3\times 2} \\ &= a^6 \end{aligned}$$

Example 2.2
Simplify $\left(g^{-2}\right)^5$
Solution

$$\begin{aligned} \left(g^{-2}\right)^5 &= g^{-2\times 5} \\ &= g^{-10} \end{aligned}$$

Example 2.3
Simplify $(mn^2)^4$
Solution

$$\begin{aligned} (mn^2)^4 &= m^4(n^2)^4 \\ &= m^4 n^{2\times 4} \\ &= m^4 n^8 \end{aligned}$$

Example 2.4
Simplify $(-c)^2$
Solution

$$\begin{aligned} (-c)^2 &= (-1)^2 \times c^2 \\ &= 1 \times c^2 \\ &= c^2 \end{aligned}$$

Example 2.5
Simplify $(-u^2)^5$
Solution

$$\begin{aligned} (-u^2)^5 &= (-1)^5 \times u^{2\times 5} \\ &= -1 \times u^{10} \\ &= -u^{10} \end{aligned}$$

Exercise B

Simplify the following

1.)	$2a \times 3a^2$	16.)	$2a \times (3a)^2$
2.)	$\sqrt[3]{2^6}$	17.)	2^{-2}
3.)	$\left(25a^2\right)^{\frac{1}{2}}$	18.)	$(2a)^{-1}$
4.)	10^{-2}	19.)	$3a^{-2}$
5.)	$\left(\frac{1}{9}\right)^{-1}$	20.)	$3^{\frac{1}{2}} \times 3^{\frac{3}{2}}$
6.)	$2a{-}1 \times 3a^2$	21.)	$2a^{-1} \times (3a)^2$
7.)	$125^{-\frac{2}{3}}$	22.)	$16^{-\frac{3}{4}}$
8.)	$4^{-\frac{3}{2}}$	23.)	$\frac{1}{3^{-2}}$
9.)	$64^{-\frac{5}{6}}$	24.)	$2x^{\frac{1}{2}} \times 3x^{-\frac{5}{2}}$
10.)	$4a^3b \times 3ab^{-2}$	25.)	$\sqrt{(125^2)^{-\frac{3}{2}}}$
11.)	$27^{\frac{1}{3}}$	26.)	$9^{\frac{1}{2}}$
12.)	$2^{-2} \times 2^3$	27.)	$\sqrt{3^4}$
13.)	$3^{\frac{1}{2}} \times 3^{-\frac{3}{2}}$	28.)	$2^{\frac{1}{2}} \times 2^{\frac{3}{2}}$
14.)	$2a \times 3q^{-2}$	29.)	$\sqrt[3]{4^{15}}$
15.)	$\left(\frac{16}{9}\right)^{-\frac{3}{2}}$	30.)	$(2x)^{\frac{1}{2}} \times (2x^3)^{\frac{3}{2}}$

WEEK THREE
INDICIAL EQUATIONS

Indicial equation with unknown base

Example 3.1
Solve $x^{\frac{1}{3}} = 4$

Solution

$$\begin{aligned} x^{\frac{1}{3}} &= 4 \\ x^{\frac{1}{3}\times\frac{3}{1}} &= 4^{1\times\frac{3}{1}} \\ x &= 4^3 \\ x &= 64 \end{aligned}$$

In general to solve indicial equation with unknown base, multiply both powers by the value that will make the power of the unknown one

Example 3.2
Solve for a in $2a^{-\frac{1}{2}} = -14$

Solution

$$\begin{aligned} 2a^{-\frac{1}{2}} &= -14 \\ a^{-\frac{1}{2}} &= -7 \\ a^{-\frac{1}{2}\times-\frac{2}{1}} &= (-7)^{-\frac{2}{1}} \\ a &= (-1)\times 7^{-2} \\ a &= 1 \times \frac{1}{49} \\ a &= \frac{1}{49} \end{aligned}$$

Indicial Equation with unknown index

To be able to solve this, not the following

$$\begin{aligned} \text{If} \quad x^a &= x^b \\ \text{then} \quad a &= b \end{aligned}$$

Example 3.3
If $5^x = 25$ find the value of x
Solution

$$\begin{aligned} 5^x &= 25 \\ 5^x &= 5^2 \\ x &= 2 \end{aligned}$$

In general, to solve Indicial equation with unknown index; make the base on both sides of the equality to be the same. Then equate their index

Example 3.4
Solve $4^{c-1} = 64$
Solution

$$\begin{aligned} 4^{c-1} &= 64 \\ 2^{2(c-1)} &= 2^5 \\ 2(c-1) &= 5 \\ 2c-2 &= 5 \\ 2c &= 5+2 \\ 2c &= 7 \\ c &= \frac{7}{2} \end{aligned}$$

Exercise C

Solve the following equations

1.) $5^m = 625$	2.) $3^m = \frac{1}{7}$	3.) $4^m = 32$
4.) $2^x = 8$	5.) $3^{2x} = 243$	6.) $5^{3x} = \frac{1}{625}$
7.) $x^{\frac{1}{2}} = 2$	8.) $x^{\frac{1}{3}} = 3$	9.) $a^{-1} = 2$
10.) $a^{-2} = 9$	11.) $2x^3 = 54$	12.) $x^{-\frac{1}{2}} = 5$
13.) $n^{-\frac{2}{3}} = 9$	14.) $2r^{-3} = -16$	15.) $5x = 40x^{-\frac{1}{2}}$
16.) $9^x = 27$		

WEEK FOUR
LOGARITHM OF NUMBERS

Logarithm - Indices relationship

Consider

$$10^2 = 100$$

This above is read as "10 to the power 2 is 100" and it means that "10 multiplying itself twice results to 100"

We introduce the term "log" and with it we can re-write the above as

$$\log_{10} 100 = 2$$

This means that "The number of times 10 would multiply itself to be 100 is 2" and is read as "logarithm (or log) to base 10 of 100 is 2"

The connection between indices and logarithm is that

$$\text{If} \quad x^a = b$$
$$\text{Then} \quad \log_x b = a$$

This is read as log to the base x of b is a

$$\text{If} \quad 10^3 = 1000$$
$$\text{Then} \quad 1000 = 3$$

$$\text{If} \quad 2^5 = 32$$
$$\text{Then} \quad \log_2 32 = 5$$

Logarithm to the base 10 are called **common logarithm**. logarithm used in calculations are normally expressed in base 10

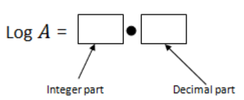
Laws of Logarithm

1. $\log_x (A \times b) = \log_x A + \log_x B$
2. $\log_x (A \div B) = \log_x \left(\frac{A}{B}\right) = \log_x A - \log_x B$
3. $\log_x A^b = b \log_x A$

Use of Logarithm Table

Log tables helps us to find the common logarithms of number which we cant determine through outright calculation say $\log_{10} 3.2$

The logarithm of any number consist of two parts called the characteristic (or integer) and the mantissa (or decimal). That is to say that



The characteristics part is obtained by subtracting 1 from the number of digits in the integral part of the number whose logarithm is to be determined

Examples
Given $\log 399$ The characteristics is 2

Given $\log 4.21$; The characteristics is 0

Given $\log 56.25$; The characteristics is 1

The mantissa part of the logarithm is read from the logarithm table. Thus for the number 399 in determining $\log 399$, we look up for the first two digits 39 in the left hand column of the log table and follow the line across till we come under the third digit 9. Here we get 6010. Remember the mantissa is always decimal so this is .6010. Now joining the characteristics with the mantissa part we obtain

$$\log 399 = 2.6010$$

Example 4.1 Use the table to find $\log 3.7$
Solution

The characteristics is zero. Since the number of the whole number digit is 1. Subtracting one we have zero.

The mantissa part is obtained from the table by looking up the first two digits 37 (which is the only digit) in the left hand column of the log table under 0 which gives us 5682. Hence

$$\log 3.7 = 0.5682$$

Exercise D

With the aid of log table find the following

1.) $\log 75.12$	8.) $\log 4137$	15.) $\log 208.5$
2.) $\log 40.02$	9.) $\log 100.6$	16.) $\log 9$
3.) $\log 50$	10.) $\log 5.6$	17.) $\log 3$
4.) $\log 234.98$	11.) $\log 7$	18) $\log 2$
5.) $\log 5.136$	12.) $\log 5136$	19.) $\log 51.36$
6.) $\log 513600$	13.) $\log 8.403$	20.) $\log 840.3$
7.) $\log 8403000$	14.) $\log 84030$	

ANTILOGARITHM OF NUMBERS

Anti logarithm is logarithm in reverse. That is to say

If
$$\log 399 = 2.6010$$
Then
$$399 = \text{anti-}\log 2.6010$$
In fact in general

If
$$\log x = y$$
Then
$$x = \text{anti-}\log y$$

Example 4.2

Find the antilog of 1.5775
Solution
The fractional part of 1.5775 is .5775
.5775 in the antilog table gives 3780 (That is .57 under 7 difference 5). 1 is added to the integer part which

results to 2. The 2 indicates that they are two digits before the decimal point. so the

$$\text{anti-log of } 1.5775 \ = 37.80$$

Exercise E
Find the anti-log of the following numbers

1.) 0.7142	2.) 1.7142	3.) 6.7142
4.) 3.7142	5.) 0.5682	6.) 2.413
7.) 2.1814	8.) 4.2105	9.) 1.1091
10. 3.4485	11.) 2.0088	12.) 1.5638

WEEK FIVE
MULTIPLICATION AND DIVISION WITH LOGARITHM

The logarithm/antilogarithm table makes multiplication and divisions of numbers involving decimals easier when the calculator is not available or allowed

Example 5.1
Compute 2.143×3.092
Solution
Consider using logarithm on the numbers we have that

$$\begin{aligned}\log(2.413 \times 3.092) &= \log 2.413 + \log 3.092\\ &= 0.3825 + 0.4902\\ &= 0.8727\end{aligned}$$

If

$$\log(2.413 \times 3.092) = 0.8727$$

Then

$$2.413 \times 3.092 = \text{anti-log } 0.8727$$

$$2.413 \times 3.092 = 7.459$$

Using the table format 2.143×3.092 will be computed as

No	Log
2.413	0.3825
3.092	+ 0.4902
2.143×3.092	0.8727
7.459	$\Longleftarrow$

Hence $2.143 \times 3.092 = 7.459$

Example 5.2
Determine the value of 1200×85.25
Solution
Using the table format 1200×85.25 will be computed as

No	log
1200	3.0792
85.25	− 1.9307
$1200 \div 85.25$	1.1485
14.08	$\Longleftarrow$

hence $1200 \times 85.25 = 14.08$
Example 5.3
Compute 2.938^2

Solution

No	Log	
2.938	0.4681	
2.938^2	0.4681×3	1.4043
25.37	$\Longleftarrow$	$\Longleftarrow$

Hence $2.938^2 = 25.37$

Example 5.4
Evaluate $\sqrt[3]{350}$

Solution

No	Log	
350	2.5441	
$350^{\frac{1}{3}}$	$2.5441 \div 3$	0.8480
7.047	$\Longleftarrow$	$\Longleftarrow$

Hence $350^{\frac{1}{3}} = 7.047$

Exercise F

Use tables to work out the following. Do a rough check in every case

1.) 2.415×3.092	18.) $9.475 \div 6.13$
2.) $8.735 \div 3.909$	19.) 3.338×2.074
3.) $46.31 \div 8.742$	20.) 45.34×16.21
4.) 26.52×9.184	21.) 5.037^2
5.) 7.214^3	22.) 2.539^5
6.) $\sqrt[10]{2.882}$	23.) $\frac{86.23 \times 4058}{913.6}$
7.) $2.96^2 \times 8.542$	24.) $\left(\frac{95.32}{8.971}\right)^2$
8.) 1.084^{10}	25.) $3.95^3 \times 62.5$
9.) $5.836^2 \times 1.283^3$	26.) $\left(\frac{403.9}{79.62}\right)^3$
10.) $29.3 \times \sqrt[5]{3.87}$	27.) $\frac{28.61 \times 74.23}{355.9 \times 2.547}$
11.) $\frac{943}{11.64 \times 7.189}$	32.) $\frac{(17.2)^2 \times 4.93}{\sqrt[3]{6750000}}$
12.) 3.802×2.09	28.) 98.15×7.264
13.) $176.3 \div 92.48$	29.) 61.03^2
14.) $\sqrt[3]{26.21}$	30.) $5.932 \times 8.164 \times 18.51$
15.) $\sqrt[3]{3.172 \times 19.86}$	31.) $\sqrt[3]{\frac{1067}{29.4}}$
16.) $\sqrt[3]{\frac{218 \times 37.2}{95.43}}$	32.) $\frac{315 \times 95.47}{456.2 \times 31.88}$
17.) $\sqrt[3]{\left(\frac{38.32 \times 2.964}{8.637 \times 6.285}\right)^2}$	

WEEK 7
MODULAR ARITHMETIC

In the eastern part of Nigeria days are read as ”eke”, ”orie”, ”afor” and ”Nkwo”. This is called the 4 -

market days system. In fact calendars particularly the ones made in/for the southeastern states have the 4-market days system inscribed into them

Now consider this question:

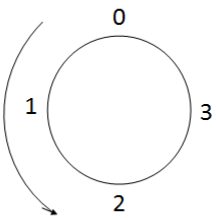
If today is 0rie, what market day will it be in 205 days time?

Before going on to answer the above question, let us develop some facts about the question

It is easy to see that in five days time (if todays is orie) it going to be
Afor
Each market day reappears every 4 days hence the days are in a 4 day cycle

Let's use numbers to represent the 4-market days. Thus;

0 - Eke
1 - Orie
2 - Afor
3 - Nkwo



Since today is orie $\Longrightarrow$ 1. In the next 5 days, counting the next five numbers in the circle stops at 2 $\Longrightarrow$ (Which represents) Afor.

This can be written as

$$1 + 5 \Longrightarrow 2$$

Also
Since today is Orie $\Longrightarrow$ 1, In the next 12 days (Counting the next 12 numbers in the circle), it will stop at 1 $\Longrightarrow$ orie.

This can be written as

$$1 + 12 \Longrightarrow 1$$

Infact from the diagram

$$2 + 3 \Longrightarrow 1$$
$$0 + 10 \Longrightarrow 2$$
$$0 + 4 \Longrightarrow 0$$
$$1 + 4 \Longrightarrow 1$$
$$3 + 4 \Longrightarrow 3$$

The numbers recycles in 4s, Hence adding of 4s has no real effect on the outcome.

That is
$1 + 4 \Longrightarrow 1$ and $1 + 4 + 4 \Longrightarrow 1$

This kind of arithmetic where we add positive integers by rotating in number cycles is called **modular arithmetic**.

Example 7.1

$$6 = 2(\mod 4)$$

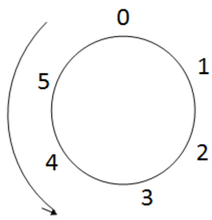
(This means starting at zero and rotating 6 around 4 will stop at the number 2)
(Another way to look at it is; The remainder of $6 \div 4$ is 2)

Example 7.2
Reduce 55 to its simplest form in (a) Modulo 3 (b) Modulo 4 (c) Modulo 5 (d) Modulo 6

Solution
(a) $55 = 1$ (mod 3)
(b) $55 = 3$ (mod 4)
(c) $55 = 0$ (mod 5)
(d) $55 = 1$ (mod 6)

Exercise G

1. Use the number cycle in the figure below to find the values of the following



   (a) $2 + 10 =$
   (b) $5 + 5 =$
   (c) $2 + 15 =$
   (d) $3 + 22 =$

2. A toy car starts at a point O and runs around a circular track of $2m$. Hoe far is the car from its starting point along the track when it has gone (a) $6m$ (b) $11m$ (c) $12m$ (d) $15m$ (e) $18m$ (f) $21m$?

3. Reduce 35 to its simplest form in (a) Modulo 6 (b) Modulo 4 (c) Modulo 5 (d) Modulo 3

Addition in Modular Arithmetic

Example 7.3
Add the following

(a) $3 + 4$ (mod 5) (b) $5 + 7$ (mod 3) (c) $2 + 6$ (mod 4)

Solution
(a)

$$\begin{aligned} 3 + 4 \pmod 5 &= 7 \pmod 5 \\ &= 2 \pmod 5 \end{aligned}$$

(b)

$$\begin{aligned} 5 + 7 \pmod 3 &= 12 \pmod 3 \\ &= 0 \pmod 3 \end{aligned}$$

(c)

$$\begin{aligned} 2 + 6 \pmod 4 &= 8 \pmod 4 \\ &= 0 \end{aligned}$$

Addition Modulo Table

Constructing addition modulo 3 table we have

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Exercise H

1. Find the following in their simplest form in modulo 4

   (a) 15 (b) 32 (c) 62 (d) 47
   (e) 78 (f) 82 (g) 17 (h) 38
   (i) 56 (j) 102

2. Find the following additions in modulo 5

   (a) $3 \oplus 9$ (b) $9 \oplus 9$ (c) $5 \oplus 8$
   (d) $8 \oplus 17$ (e) $13 \oplus 7$ (f) $6 \oplus 6$
   (g) $28 \oplus 39$ (h) $41 \oplus 52$ (i) $78 \oplus 27$

3. Calculate in the given moduli

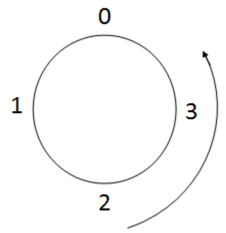
   (a) $3 \oplus 4 \pmod 2$ (b) $12 \oplus 9 \pmod 4$
   (c) $8 \oplus 5 \pmod 3$ (d) $17 \oplus 8 \pmod 9$
   (e) $9 \oplus 14 \pmod 7$ (f) $54 \oplus 25 \pmod 5$
   (g) $49 \oplus 28 \pmod 8$ (h) $72 \oplus 8 \pmod 6$
   (i) $67 \oplus 38 \pmod 8$ (j) $82 \oplus 4 \pmod 3$

4. Construct the addition modulo 5 table

Subtraction in Modulo Arithmetic

Moduli of Negative numbers

Example 7.4
$0 \ominus 3 \pmod 4 = -3 \pmod 4$



- Start at 0 and move in an anti-clock wise direction (as shown in the diagram above) 3 places. The result is 1. Hence

$$0 \ominus 3 \pmod 4 = -3 \pmod 4 = 1$$

$1 \ominus 2 \pmod 4 = -1 \pmod 4 =?$

- Start at 1 and move 2 places in the anti-clock wise direction. The result is 3. Hence

$$\begin{aligned} 1 \ominus 2 \pmod 4 &= -1 \pmod 4 \\ &= 3 \end{aligned}$$

Another way to obtain results of moduli of negative numbers is to write the numbers as the addition of two numbers of which one is the -ve multiple of the modulli and the other number is a +ve number less than the moduli. Also note the following relation

$$a + b \pmod x = a \pmod x + b \pmod x$$

Example 7.5
$-3 \pmod 4$

$$\begin{aligned} -3 \pmod 4 &= -4 + 1 \pmod 4 \\ &= -4 \pmod 4 + 1 \pmod 4 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

Example 7.6
$-1 \pmod 9$

$$\begin{aligned} -1 \pmod 9 &= -4 + 3 \pmod 4 \\ &= 3 \text{ (mod} \end{aligned}$$

Example 7.7
$5 \ominus 11 \pmod 5$

$$\begin{aligned} 5 \ominus 11 \pmod 5 &= 5 - 6 \pmod 5 \\ &= -6 \pmod 5 \\ &= -10 + 4 \pmod 5 \\ &= 4 \text{ (mod } 5 \end{aligned}$$

Exercise I

1. Find the simplest positive form of each of the following numbers in modulo 5

   (a) -9 (b) -33 (c) -104
   (d) -32 (e) -35 (f) -63
   (g) -72 (h) -34 (i) -75
   (j) -256

2. Find the simplest form of the following in the given moduli

   (a) $-5 \pmod 6$ (b) $-17 \pmod{10}$
   (c) $-23 \pmod 5$ (d) $-9 \pmod 7$
   (e) $-52 \pmod{11}$ (f) $-11 \pmod 3$
   (g) $-17 \pmod 2$ (h) $-56 \pmod{13}$
   (i) $-50 \pmod 4$ (j) $-75 \pmod 7$

3. Simplify the following

   (a) $2 \ominus 5 \pmod 4$ (b) $8 \ominus 10 \pmod 2$
   (c) $18 \ominus 28 \pmod 5$

Multiplication in modular arithmetic

Example 7.8
Simplify $2 \otimes 3 \pmod 4$

$$\begin{aligned} 2 \otimes 3 \pmod 4 &= 6 \pmod 4 \\ &= 2 \pmod 4 \end{aligned}$$

To make multiplication of big numbers easier note the relation below

$$a \otimes b \pmod x = a \pmod x \times b \pmod x$$

Example 7.9

Evaluate $15 \otimes 26 \pmod 5$

$15 = 0 \pmod 5$
$26 = 1 \pmod 5$
Hence

$$\begin{aligned} 15 \otimes 26 \pmod 5 &= 0 \otimes 1 \pmod 5 \\ &= 0 \pmod 5 \end{aligned}$$

Division in Modular arithmetic

(to be discussed in class)

Exercise J

1. Find the value of the following, mod 4

   (a) $5 \otimes 7$ (b) $5 \otimes 15$ (c) $13 \otimes 9$
   (d) $23 \otimes 18$ (e) 6×73 (f) $21 \otimes 65$

2. Find the values in the moduli written besides them

   (a) $16 \otimes 7 \pmod 5$ (b) $31 \otimes 15 \pmod 7$
   (c) $27 \otimes 4 \pmod 7$ (d) $21 \otimes 18 \pmod{10}$
   (e) $8 \otimes 25 \pmod 3$ (f) $80 \otimes 29 \pmod 7$

3. Construct the multiplication moduli 5 table

WEEK EIGHT
SURDS

Numbers such as 5, $2\frac{1}{3}$, 0.37, 0.6, $\sqrt{49}$ can be written as fraction. These kind of numbers are called rational numbers. because we can write them as

$5 = \frac{5}{1}$; $2\frac{1}{3} = \frac{7}{3}$; $0.37 = \frac{37}{100}$; $0.6 = \frac{6}{10}$; $\sqrt{49} = \pm\frac{7}{1}$

They are numbers that cannot be written as exact fraction. An example is the π where

$$\pi = 3.141592...$$

These kind of numbers are referred to as Irrational Numbers
Other examples of irrational numbers are $\sqrt{2} = 1.414...$, $\sqrt{3} = 1.7320...$ etc
Irrational numbers in the form

$$a\sqrt{b}$$

where b is not a perfect square is called a Surd.
Examples of surds are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{10}$, $\sqrt{12}$, $\sqrt{13}$ etc

Laws of Surd

1. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
2. $\sqrt{\frac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Note that
$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$;$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$

Simplification of Surds
Simplification of surds involves making the number under the square root sign to be as small as possible. This is achieved by expressing the number under the square root sign as a product of two factors, one of which is a perfect square

Example 8.1
Simplify (a) $\sqrt{45}$ (b) $3\sqrt{50}$
Solution

(a)

$$\begin{aligned} \sqrt{45} = \sqrt{9 \times 5} &= \sqrt{9} \times \sqrt{5} \\ &= 3 \times \sqrt{5} \\ &= 3\sqrt{5} \end{aligned}$$

(b)

$$\begin{aligned} 3\sqrt{50} = 3 \times \sqrt{25 \times 2} &= 3 \times 5 \times \sqrt{2} \\ &= 15\sqrt{2} \end{aligned}$$

Exercise K
Simplify the following
(a) $\sqrt{20}$ (b) $\sqrt{32}$ (c) $\sqrt{48}$ (d) $\sqrt{75}$ (e) $\sqrt{72}$ (f) $\sqrt{24}$
(g) $\sqrt{200}$ (h) $\sqrt{150}$ (i) $\sqrt{98}$ (j) $\sqrt{84}$

Example 8.2
Express $2\sqrt{6}$ as the square root of a single number
Solution
$2\sqrt{6} = \sqrt{4} \times \sqrt{6} = \sqrt{4 \times 6} = \sqrt{24}$

Class Exercise L

Express the following as the square root of a single number
(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) $2\sqrt{2}$ (d) $5\sqrt{7}$ (e) $3\sqrt{10}$ (f) $2\sqrt{7}$ (g) $4\sqrt{6}$ (h) $3\sqrt{8}$ (i) $2\sqrt{11}$ (j) $5\sqrt{5}$

Like Surds
Two or more surds are said to be like surds if the numbers under the square root sign are the same. Example; $\sqrt{2}$, $2\sqrt{2}$, $\frac{1}{5}\sqrt{2}$ are like surds.

Addition/Subtraction of Surds
Two or more surds can be added/subtracted only if they are like surds. Remember, before adding/subtracting two more surds, the individual surds should first be simplified.

Example 8.3
Simplify $5\sqrt{2} - 2\sqrt{2}$
Solution
$5\sqrt{2} - 2\sqrt{2} = (5-2)\sqrt{2} = 3\sqrt{2}$
Example 8.4
Simplify $\frac{2}{3}\sqrt{5} - 4\sqrt{5}$
Solution

$$\begin{aligned}\frac{2}{3}\sqrt{5} - 4\sqrt{5} &= \left(\frac{2}{3} - 4\right)\sqrt{5}\\ &= \frac{2-12}{3}\sqrt{5}\\ &= -\frac{10}{3}\sqrt{5}\end{aligned}$$

Example 8.5
Simplify $\sqrt{12} + \sqrt{3}$
Solution

$$\begin{aligned}\sqrt{12} + \sqrt{3} &= 2\sqrt{3} + \sqrt{3}\\ &= 3\sqrt{3}\end{aligned}$$

Exercise N
Simplify the following

1. $3\sqrt{2} - \sqrt{18}$
2. $\sqrt{175} - 4\sqrt{7}$
3. $2\sqrt{8} - 3\sqrt{32} + 4\sqrt{50}$
4. $2\sqrt{54} + \sqrt{24} - \sqrt{216}$
5. $3\sqrt{125} - 5\sqrt{20} + 3\sqrt{80}$
6. $\sqrt{60} - \sqrt{375} + \sqrt{135}$
7. $2\sqrt{135} - 2\sqrt{60} + \sqrt{15} - \sqrt{240}$
8. $6 + \sqrt{27} + \sqrt{75}$
9. $\sqrt{52} - \sqrt{117} + 4\sqrt{13}$
10. $\sqrt{224} - \sqrt{126} - \sqrt{56}$

Multiplication of Surds
When two or more surds are multiplied together, they should first be simplified, if possible. Then multiply whole number with whole numbers and surds with surds. Also remember that

$$\sqrt{a} \times \sqrt{a} = a$$

Example 8.6
Simplify $\sqrt{27} \times \sqrt{50}$
Solution

$$\begin{aligned}\sqrt{27} \times \sqrt{50} &= 3\sqrt{3} \times 5\sqrt{2}\\ &= 3 \times 5 \times \sqrt{3} \times \sqrt{2}\\ &= 15 \times \sqrt{3 \times 2} = 15\sqrt{6}\end{aligned}$$

Exercise O
Simplify the following

1. $\sqrt{5} \times \sqrt{10}$
2. $\sqrt{8} \times \sqrt{2}$
3. $\sqrt{12} \times \sqrt{3}$
4. $\sqrt{30} \times \sqrt{5}$
5. $\sqrt{32} \times \sqrt{12}$
6. $(\sqrt{3})^5$
7. $(2\sqrt{7})^2$
8. $\sqrt{5} \times \sqrt{24} \times \sqrt{30}$
9. $\sqrt{6} \times \sqrt{8} \times \sqrt{10} \times \sqrt{12}$
10. $(2\sqrt{3})^3$

Fractional Surds
These are fraction that contains surd either in the numerator or denominator or both. Examples are $\frac{\sqrt{2}}{3}$, $\frac{5}{\sqrt{7}}$, $\frac{3\sqrt{8}}{2\sqrt{7}}$

Conjugate of a Surd $a\sqrt{b}$
The conjugate of a surd $a\sqrt{b}$ is another surd in which their multiplication results in a rational number. Generally the conjugate of $a\sqrt{b}$ is $\sqrt{b}$. For Example the conjugate of $3\sqrt{2}$ is $\sqrt{2}$ because

$$\begin{aligned}3\sqrt{3} \times \sqrt{2} &= 3 \times \sqrt{2} \times \sqrt{2}\\ &= 3 \times 2\\ &= 6\end{aligned}$$

Rationalization of Denominator
This simply means to convert the denominator of a fractional surd into a rational number. To do this, we multiply the numerator and denominator of the fraction by the conjugate of the denominator.
Example 8.8
Rationalize $\dfrac{6}{\sqrt{3}}$
Solution

$$\begin{aligned}\frac{6}{\sqrt{3}} &= \frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{6\sqrt{3}}{3}\\ &= 2\sqrt{3}\end{aligned}$$

Example 8.9
Rationalize $\frac{7}{\sqrt{18}}$
Solution

$$\begin{aligned}\frac{7}{\sqrt{18}} &= \frac{7}{3\sqrt{2}}\\ &= \frac{7}{3\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{7\sqrt{2}}{3(2)}\\ &= \frac{7\sqrt{2}}{6}\end{aligned}$$

Example 9.0
Rationalize $\frac{\sqrt{5}}{\sqrt{2}}$
Solution

$$\begin{aligned}\frac{\sqrt{5}}{\sqrt{2}} &= \frac{\sqrt{5}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Exercise P
Simplify the following by rationalizing the denominators

1. $\frac{2}{\sqrt{2}}$
2. $\frac{6}{\sqrt{2}}$
3. $\frac{4}{\sqrt{8}}$
4. $\frac{15}{\sqrt{3}}$
5. $\frac{2\sqrt{3}}{\sqrt{6}}$
6. $\frac{30}{\sqrt{75}}$
7. $\frac{30}{\sqrt{72}}$
8. $\frac{3\sqrt{2}}{\sqrt{10}}$
9. $\frac{\sqrt{3} \times \sqrt{18} \times \sqrt{39}}{\sqrt{24} \times \sqrt{26}}$

Binomial Surds
An expression may contain two terms, which cannot be simplified further. For eg, $6 - \sqrt{5}$, $3\sqrt{2} + \sqrt{3}$, $2\sqrt{3} + 4$. If one or both of the terms contain a surd, we call this a binomial surd expression.

Multiplication of Binomial Surds
When multiplying two binomial surds, use the normal algebraic expansion

$$(a + b)(c + d) = ac + ad + bc + bd$$

Example 9.1
Expand and simplify $(3\sqrt{5} + 2)(\sqrt{5} + 3)$
Solution

$$\begin{aligned}(3\sqrt{5} + 2)(\sqrt{5} + 3) &= 3(5) + 9\sqrt{5} + 2\sqrt{5} + 6\\ &= 15 + 11\sqrt{5} + 6\\ &= 21 + 11\sqrt{5}\end{aligned}$$

Example 9.2
Expand and simplify $2\sqrt{5}(3\sqrt{5} - 2\sqrt{2})$
Solution

$$\begin{aligned}2\sqrt{5}(3\sqrt{5} - 2\sqrt{2}) &= 6(5) - 4\sqrt{10}\\ &= 30 - 4\sqrt{10}\end{aligned}$$

Example 9.3
Expand and simplify $(2\sqrt{2} + \sqrt{5})^2$
Solution

$$\begin{aligned}(2\sqrt{2} + \sqrt{5})^2 &= (2\sqrt{2} + \sqrt{5})(2\sqrt{2} + \sqrt{5})\\ &= 4(2) + 2\sqrt{10} + 2\sqrt{10} + 5\\ &= 8 + 4\sqrt{10} + 5\\ &= 13 + 4\sqrt{10}\end{aligned}$$

Exercise Q
Simplify the following

1. $\sqrt{2}(\sqrt{2} + \sqrt{6})$
2. $(\sqrt{5} + \sqrt{15})(2\sqrt{3} - 1)$
3. $(\sqrt{6} + 2\sqrt{3})^2$
4. $(3\sqrt{2} - \sqrt{5})^2$

Conjugate of a binomial surd
The conjugate of a binomial surd $a + \sqrt{d}$ is $a - \sqrt{d}$ and vice versa. **Example 9.4**
Simplify the following by rationalizing the denominators

(a) $\frac{2}{3\sqrt{5} + 4}$

(b) $\frac{6}{2\sqrt{2} - 1}$

(c) $\frac{2\sqrt{3} + 2}{2\sqrt{3} - 2}$

Solution

(a)

$$\begin{aligned}\frac{2}{3\sqrt{5} + 4} &= \frac{2}{3\sqrt{5} + 4} \times \frac{3\sqrt{5} - 4}{3\sqrt{5} - 4}\\ &= \frac{2(3\sqrt{5} - 4)}{(3\sqrt{5} + 4)(3\sqrt{5} - 4)}\\ &= \frac{6\sqrt{5} - 8}{9(5) - 12\sqrt{5} + 12\sqrt{5} - 16}\\ &= \frac{6\sqrt{5} - 8}{45 - 6}\\ &= \frac{6\sqrt{5} - 8}{29}\end{aligned}$$

(b)

$$
\begin{aligned}
\frac{6}{2\sqrt{2}-1} &= \frac{6}{2\sqrt{2}-1} \times \frac{2\sqrt{2}+1}{2\sqrt{2}+1} \\
&= \frac{6(2\sqrt{2}+1)}{(2\sqrt{2}-1)(2\sqrt{2}+1)} \\
&= \frac{12\sqrt{2}+6}{4(2)+2\sqrt{2}-2\sqrt{2}-1} \\
&= \frac{12\sqrt{2}+6}{8-1} \\
&= \frac{12\sqrt{2}+6}{7}
\end{aligned}
$$

(c)

Exercise R
Simplify the following by rationalizing the denominator

1. $\dfrac{1}{2-\sqrt{3}}$
2. $\dfrac{4}{3+\sqrt{7}}$
3. $\dfrac{\sqrt{5}}{\sqrt{15}-\sqrt{10}}$
4. $\dfrac{3\sqrt{7}}{5-\sqrt{7}}$
5. $\dfrac{\sqrt{6}+\sqrt{2}}{\sqrt{6}-\sqrt{2}}$
6. $\dfrac{2\sqrt{3}-\sqrt{5}}{2\sqrt{3}+\sqrt{5}}$
7. $\left(\dfrac{\sqrt{3}\sqrt{2}}{\sqrt{3}+\sqrt{2}}\right)^2$